Chapter 4 Part 1 Notes
4.1 – Estimating Roots

**Goal:** to explore decimal representations of different roots of numbers

**Toolkit:**
- Finding a square root
- Finding a cube root
- Multiplication
- Estimating

**Main Ideas:**

**Definitions:**

Radical: an expression consisting of a radical sign, a radicand, and an index.

Perfect squares and cubes to memorize: \( \sqrt{4} = 2 \), \( \sqrt{9} = 3 \), \( \sqrt{16} = 4 \), \( \sqrt{25} = 5 \), \( \sqrt{36} = 6 \), \( \sqrt{49} = 7 \), \( \sqrt{64} = 8 \), \( \sqrt{81} = 9 \), \( \sqrt{8} = 2 \), \( \sqrt{27} = 3 \), \( \sqrt{64} = 4 \), \( \sqrt{125} = 5 \)

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Ex 1) Evaluate the following radicals, identify the radicand and index for each:

a) \( \sqrt{16} = 4 \)
   
   \[ 4 \times 4 = 16 \]

   **Radicand:** 16  
   **Index:** 2

b) \( \sqrt{64} = 4 \)

   \[ 4 \times 4 \times 4 = 64 \]

   **Radicand:** 64  
   **Index:** 3

*If no index is written the index is a 2.*

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Ex 2) Estimate the value of \( \sqrt{20} \) to one decimal place.

**Step 1:** Find the two perfect squares that are closest to the radicand you are looking for (one that is lower and one that is higher).

\[ \sqrt{16} \quad \sqrt{25} \]

\[ \frac{16}{4} \quad \frac{25}{5} \]

**Step 2:** Find which of the two perfect squares is closest to your radicand; this will determine the decimal point of your root.

\[ 20 - 16 = 4 \]

\[ \sqrt{20} \text{ is closer to } \sqrt{16} \text{ so the root is closer to } 4. \]

\[ \sqrt{25} - 20 = 5 \]

\[ \text{Your estimate: } 4.4 \]

Evaluate \( \sqrt{20} \), how close was your estimate?

\[ \sqrt{20} = 4.47 \]
Ex 3) Estimate the value of \( \sqrt[3]{16} \)

**Step 1:** Find the two perfect cubes that are closest to the radicand you are looking for.

\[
\begin{align*}
3^3 &= 27 \
2^3 &= 8 \\
\sqrt[3]{16} &\approx 2.4
\end{align*}
\]

**Step 2:** Find which of the two perfect cubes is closest to your radicand.

\[
16 - 8 = 8 \quad \rightarrow \quad 3\sqrt[3]{16} \text{ is closest to } 2\sqrt[3]{8}
\]

Evaluate \( \sqrt[3]{16} \), how close was your estimate?

\[
\sqrt[3]{16} \approx 2.5
\]

Ex 4) Estimate the value of \( \sqrt[3]{-32} \)

\[
(-8)(-8)(-8) = (-)
\]

\[
\begin{align*}
3\sqrt[3]{-32} &\quad \text{is significantly closer to} \\
3\sqrt[3]{-27} &\quad \text{since the root is closer to} \\
\sqrt[3]{-4} &\quad \text{so the root is closer to} \\
-3.2 &\quad \text{hence,} \\
3\sqrt[3]{-32} &\approx -3.2
\end{align*}
\]

Ex 5) Evaluate \( \sqrt{0.64} \).

*If radicand has 2 decimal places, then root has one decimal place.*

We know \( \sqrt{64} = 8 \)

So, \( \sqrt{0.64} = 0.8 \)

Ex 6) Evaluate \( \sqrt{0.0196} \).

*If radicand has 4 decimal places, then root has two.*

We know \( \sqrt{196} = 14 \)

So, \( \sqrt{0.0196} = 0.14 \)

Ex 7) Write an equivalent form of 0.3 as a cube root.

\[0.3 \times 0.3 \times 0.3 = 0.027 \quad \rightarrow \quad \sqrt[3]{0.027} = 0.3\]

**Reflection:** How would you write 5 as a square root? A cube root? A fourth root?

- Square root: \( \sqrt{25} = 5 \)
- Cube root: \( \sqrt[3]{125} = 5 \)
- Fourth root: \( \sqrt[4]{625} = 5 \)
4.2 – Irrational Numbers

Goal: to classify real numbers, and to identify & order irrational numbers

Toolkit:
- Estimating roots
- Placing numbers on number lines
- Anything you remember about classifying Real Numbers

Main Ideas:

Natural Numbers (\(\mathbb{N}\)) Counting numbers.

Whole Numbers (\(\mathbb{W}\)) Zero, & counting#s

Integers (\(\mathbb{Z}\)) negative, counting#, zero, positive, counting#s

Rational Numbers (\(\mathbb{Q}\)) can be written as \(\frac{m}{n}\) integers (n\(\neq 0\)). Decimals: repeating (0.3) \(\frac{3}{10}\) or terminating (ending - 0.25) \(\frac{1}{4}\)

Irrational Numbers (\(\mathbb{Q}\)) Not rational! Cannot be written as a fraction. Decimals do not repeat or terminate. (\(\sqrt{2}, \\pi\))

Classifying Real Numbers

Ex1) Where do these numbers belong in the diagram of Real numbers?

\[
\begin{align*}
2 & 0.6 \quad 4\sqrt{2} \quad \frac{4}{3} \quad -8 \quad \frac{0}{\sqrt{4}} \quad -12 \quad \pi \quad 0 \quad \sqrt{16} = 4 \\
1.35 & \quad -\sqrt{125} \quad \sqrt{3} \quad \sqrt[3]{15} \quad \sqrt{\frac{4}{9}} = \frac{2}{3}
\end{align*}
\]

Real Numbers: steps “terminates”

Rational Numbers \(\mathbb{Q}\)

Irrational Numbers \(\mathbb{Q}\)
Ex2) Use a number line to order these numbers from least to greatest.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{6}$</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{11}$</td>
<td>$\sqrt{30}$</td>
<td>$\sqrt{2}$</td>
</tr>
</tbody>
</table>

between $3\sqrt{2}$ and $3\sqrt{8}$,

$\approx -1.26$ calc

$c = 3.32$ calc

$c = 2.34$ calc

$c = 1.41$ calc

Or use calc:

$3\sqrt{6} \approx 1.82$

Ex3) Is the tangent ratio for $\theta$ in each right triangle rational or irrational?

a)

$\tan \theta = \frac{opp}{adj}$

$\tan \theta = \frac{3}{y}$ Rational!

b)

$\tan \theta = \frac{opp}{adj}$

$\tan \theta = \frac{\sqrt{3}}{1}$ Irrational!

Reflection: How could you order a set of irrational numbers if you do not have a calculator?

estimate all of the decimals, then place on a number line
### 4.3A – From Entire to Mixed Radicals

**Goal:** to express an entire radical as a mixed radical

<table>
<thead>
<tr>
<th><strong>Toolkit:</strong></th>
<th><strong>Main Ideas:</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>- Understanding Radicals</td>
<td>-</td>
</tr>
<tr>
<td>- Identifying Factors of a Number</td>
<td>-</td>
</tr>
</tbody>
</table>

**Perfect Squares** - 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, ...

**Perfect Cubes** - 1, 8, 27, 64, 125, 216, ...

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**A radical sign with a number under it:** \( \sqrt[3]{28}, \sqrt[3]{64} \)

**A number written as the product of a number and a radical:** \( 3\sqrt{5}, 4\sqrt{10} \)

### Equivalent Forms:

Ex. 1)

a) \( \sqrt{16} \cdot \sqrt{9} \) is equivalent to \( \sqrt{16} \cdot \sqrt{9} \) because:

\[
\sqrt{16} = 4 \cdot 3 \quad \sqrt{27} = 3 \cdot 3
\]

\[
12 = \sqrt{12} \quad 6 = \sqrt{6}
\]

\[ \sqrt{ab} = \sqrt{a} \cdot \sqrt{b}, \text{ where } n \text{ is a natural number, and } a \text{ and } b \text{ are real numbers} \]

---

**What is an entire radical?**

**What is a mixed radical?**

**What is the Multiplication Property of Radicals?**

---

**Simplifying Square Roots**

We can simplify \( \sqrt{24} \) because 24 has a perfect square factor of 4 (hint: look at list of perfect squares)

- Re-write \( \sqrt{24} \) as a product of two factors, with the first one being the perfect square:

\[
\sqrt{24} = \sqrt{4} \cdot \sqrt{6} = 2 \sqrt{6}
\]

**Simplifying Cube Roots**

We can also simplify \( \sqrt[3]{24} \) because 24 has a perfect cube factor of 8 (hint: look at list of perfect cubes)

- Re-write \( \sqrt[3]{24} \) as a product of two factors, with the first one being the perfect cube:

\[
\sqrt[3]{24} = \sqrt[3]{8} \cdot \sqrt[3]{3} = 2 \sqrt[3]{3}
\]

**Factors of 24:** (1, 2, 3, 4, 6, 12, 24)

**Factors of 24:** (1, 2, 3, 4, 6, 12, 24)
Ex. 2) Simplify each radical: (remember your list of perfect squares and perfect cubes!)

<table>
<thead>
<tr>
<th>a) $\sqrt{80}$</th>
<th>b) $\sqrt{32}$</th>
<th>c) $\sqrt{98}$</th>
<th>d) $\sqrt{162}$</th>
<th>e) $\sqrt{108}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>perfect square factors of 80:</td>
<td>perfect square factors of 32:</td>
<td>perfect square factors of 98:</td>
<td>perfect cube factor of 162:</td>
<td>perfect cube factor of 108:</td>
</tr>
<tr>
<td>4, 10</td>
<td>4, 8</td>
<td>4, 7</td>
<td>6, 9</td>
<td>6, 9</td>
</tr>
<tr>
<td>use 16</td>
<td>use 16</td>
<td>use 16</td>
<td>use 16</td>
<td>use 16</td>
</tr>
<tr>
<td>$\sqrt{80} = \sqrt{16 \cdot 5}$</td>
<td>$\sqrt{32} = \sqrt{16 \cdot 2}$</td>
<td>$\sqrt{98} = \sqrt{49 \cdot 2}$</td>
<td>$\sqrt{162} = \sqrt{81 \cdot 2}$</td>
<td>$\sqrt{108} = \sqrt{36 \cdot 3}$</td>
</tr>
<tr>
<td>$= 4 \sqrt{5}$</td>
<td>$= 4 \sqrt{2}$</td>
<td>$= 7 \sqrt{2}$</td>
<td>$= 9 \sqrt{2}$</td>
<td>$= 6 \sqrt{3}$</td>
</tr>
</tbody>
</table>

Ex. 3) Simplify $\sqrt[4]{162}$

$\sqrt[4]{162} = \sqrt[4]{81 \cdot 2} = \sqrt[4]{9 \cdot 9 \cdot 2} = 3 \sqrt[4]{2} \leftarrow$ prime factorization... and 3 is written 4 times!

Ex. 4) Simplify $\sqrt[4]{48}$

$\sqrt[4]{48} = \sqrt[4]{8 \cdot 6} = \sqrt[4]{4 \cdot 2 \cdot 2 \cdot 3} = \sqrt[4]{2 \cdot 2 \cdot 2 \cdot 3} \leftarrow 2$ is written 4 times!

$= \sqrt[4]{2} \cdot \sqrt[4]{2} \cdot \sqrt[4]{2} \cdot \sqrt[4]{3} = 2 \sqrt[4]{3}$

Ex. 5) A cube has a volume of 128 cm$^3$. Write the edge length of the cube in simplest radical form.

$V = 128$ cm$^3$

$V_{box} = l \cdot w \cdot h$

$V_{cube} = e \cdot e \cdot e$

$V_{cube} = e^3 \rightarrow$ but $V_{cube} = 128$ cm$^3$.

$128 = e^3 \rightarrow$ cube root both sides

$e^3 = \sqrt[3]{128} \rightarrow$ largest perfect cube factor: $8$ by $64$

$e = \sqrt[3]{64} \cdot \sqrt[3]{2} = 4 \sqrt[3]{2} \cdot \sqrt[3]{2} = 2 \sqrt[3]{4}$

Reflection: How do you use the index of a radical when you simplify a radical? Use an example.

$\sqrt{18} \rightarrow$ index 2, tells us to look for a perfect square factor

$\sqrt[3]{16} \rightarrow$ index 3, " " " " " " cube factor

general $\sqrt[n]{a} \rightarrow$ index n, " " " " " " power of n factor.
4.3B – From Mixed to Entire Radicals

Goal: to express a mixed radical as an entire radical

Toolkit:
- List of Perfect Squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100...
- List of Perfect Cubes: 1, 8, 27, 64, 125, 216, ....
- Multiplication Property of Radicals \( \sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \)
- Mixed Radical...ex. \( 3 \sqrt[4]{7}, 2 \sqrt[5]{5}, 6 \sqrt[6]{6} \)
- Entire Radical......ex. \( \sqrt[10]{5}, \sqrt[3]{3} \)

Main Ideas:

Write the mixed radical \( 4\sqrt[3]{3} \) as an entire radical:

\[
\begin{align*}
4\sqrt[3]{3} &= 4 \cdot \sqrt[3]{3} \\
&= \sqrt[6]{16} \cdot \sqrt[3]{3} \\
&= \sqrt[6]{48}
\end{align*}
\]
- Use the Multiplication Property of Radicals
(re-write 4 as a radical.....think .....4 = \sqrt[6]{48}!)
- Combine these under the same radical sign and multiply
(***NOTICE...these are the opposite steps to writing an entire radical as a mixed radical)

Ex. 1) Write each as an entire radical:

a) \( 5\sqrt[2]{2} \)
\[
\begin{align*}
5\sqrt[2]{2} &= 5 \cdot \sqrt[2]{2} \\
&= \sqrt[4]{25} \cdot \sqrt[2]{2} \\
&= \sqrt[4]{50}
\end{align*}
\]

b) \( 3\sqrt[3]{3} \)
\[
\begin{align*}
3\sqrt[3]{3} &= 3 \cdot \sqrt[3]{3} \\
&= \sqrt[6]{27} \cdot \sqrt[3]{3} \\
&= \sqrt[6]{81}
\end{align*}
\]

c) \( 3\sqrt[2]{2} \)
\[
\begin{align*}
3\sqrt[2]{2} &= 3 \cdot \sqrt[2]{2} \\
&= \sqrt[4]{9} \cdot \sqrt[2]{2} \\
&= \sqrt[4]{18}
\end{align*}
\]

d) \( 2\sqrt[6]{6} \)
\[
\begin{align*}
2\sqrt[6]{6} &= 2 \cdot \sqrt[6]{6} \\
&= \sqrt[12]{36} \\
&= \sqrt[12]{48}
\end{align*}
\]

Write \( 3\sqrt[5]{2} \) as an entire radical:

First, re-write 3 as \( \sqrt[5]{3^5} \) .... \( 3 = \sqrt[3]{3 \cdot 3 \cdot 3 \cdot 3} = \sqrt[5]{243} \)

So now, \( 3\sqrt[5]{2} \) .... \( 3 = \sqrt[6]{3^5} \cdot \sqrt[5]{2} \)

Now, using the Multiplication Property of Radicals...

\[
\begin{align*}
3\sqrt[5]{2} &= 3 \cdot \sqrt[5]{2} \\
&= \sqrt[5]{3^5} \cdot \sqrt[5]{2} \\
&= \sqrt[5]{486}
\end{align*}
\]

Ex. 2) Write each as an entire radical:

a) \( 2\sqrt[5]{5} \)
\[
\begin{align*}
2\sqrt[5]{5} &= 2 \cdot \sqrt[5]{5} \\
&= \sqrt[5]{2^5} \cdot \sqrt[5]{5} \\
&= \sqrt[5]{80}
\end{align*}
\]

b) \( 4\sqrt[2]{2} \)
\[
\begin{align*}
4\sqrt[2]{2} &= 4 \cdot \sqrt[2]{2} \\
&= \sqrt[4]{2^4} \cdot \sqrt[2]{2} \\
&= \sqrt[4]{16} \cdot \sqrt[2]{2} \\
&= \sqrt[4]{32}
\end{align*}
\]
Ex. 3) Arrange the following in order from greatest to least: \(3\sqrt{5}, 2\sqrt{13}, 4\sqrt{3}, 2, 9\sqrt{2}\)

* Re-write ALL as entire radicals:

\[
\begin{align*}
3\sqrt{5} &= \sqrt{45} \\
2\sqrt{13} &= \sqrt{52} \\
4\sqrt{3} &= \sqrt{48} \\
2 &= \sqrt{4} \\
9\sqrt{2} &= \sqrt{162}
\end{align*}
\]

* Now, it is easy to arrange these greatest to least:

\[
\sqrt{162}, \sqrt{52}, \sqrt{48}, \sqrt{45}, \sqrt{4}
\]

* Finally, replace these with the original mixed radicals:

\[
9\sqrt{2}, 2\sqrt{13}, 4\sqrt{3}, 3\sqrt{5}, 2
\]

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**Reflection:** How do you use the index of a radical when you write a mixed radical as an entire radical? Use an example to help your explanation.

When you re-write the whole number, you must use the index to determine the new radicand... ex. \(2\sqrt[3]{3} \quad 2^3\sqrt[3]{3}
\[
\begin{align*}
2\sqrt[4]{3} &= \sqrt[4]{12} \\
3\sqrt[3]{3} &= \sqrt[3]{27}
\end{align*}
\]

\[
4 = 2^2 \quad 8 = 2^3
\]